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Optimal Attitude Matrix from Two Vector Measurements

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Introduction

A SIMPLE counting argument shows that a minimum of two unit-vector measurements, with two independent scalar pieces of information each, are required to determine the 3 degrees of freedom needed to specify a spacecraft's attitude. The earliest published algorithm for attitude determination from two vector measurements was Black's TRIAD algorithm [1,2], also known as the algebraic method [3]. TRIAD has been applied to both ground-based and onboard attitude determination, using either unit vectors to the sun and along the Earth's magnetic field for a coarse "sun-mag" attitude determination or unit vectors to two stars for a precise attitude determination. TRIAD is suboptimal because it ignores one piece of information from one of the unit vectors. Modern star trackers can track 5, 6, or even 50 stars at a time, creating a need for methods that can use all of the information from two or more unit vectors, weighted in some "optimal" fashion. This need had been anticipated in 1965, when Wahba proposed her famous problem to find the proper orthogonal matrix A (i.e., with determinant $+1$) that minimizes the loss function [4]

$$L(A) \equiv \frac{1}{2}\sum_i a_i |\mathbf{b}_i - A\mathbf{r}_i|^2 \tag{1}$$

where $\{\mathbf{b}_i\}$ is a set of n unit vectors measured in a spacecraft's body frame, $\{\mathbf{r}_i\}$ are the corresponding unit vectors in a reference frame, and $\{a_i\}$ are nonnegative weights. The original solutions to this problem solved for the attitude matrix A directly [5–9], but most practical applications have been based on Davenport's q method [3], which solves for the attitude quaternion [10,11]. The present paper concentrates on the special case of two observations, for which Shuster showed that his QUEST algorithm provides a closed-form quaternion solution [2]. The purpose of this paper is to present a corresponding explicit closed-form solution for the attitude matrix and a covariance analysis of the solution.

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Singular Value Decomposition for the Two-Observation Case

We can rewrite Eq. (1) using the invariance of the matrix trace under cyclic permutations as

$$L(A) \equiv \frac{1}{2}\sum_i a_i(|\mathbf{b}_i|^2 + |A\mathbf{r}_i|^2) - \sum_i a_i \mathbf{b}_i^T A\mathbf{r}_i = \left(\sum_i a_i\right) - \mathrm{trace}(AB^T) \tag{2}$$

where

$$B \equiv \sum_i a_i \mathbf{b}_i \mathbf{r}_i^T \tag{3}$$

This shows that the proper orthogonal matrix that maximizes $\mathrm{trace}(AB^T)$ is the desired optimal attitude matrix. We note that B has the singular value decomposition (SVD) [12–14]

$$B = USV^T \tag{4}$$

where U and V are orthogonal matrices and S is diagonal:

$$S = \mathrm{diag}(s_1, s_2, s_3) \tag{5}$$

with

$$s_1 \geq s_2 \geq s_3 \geq 0 \tag{6}$$

In the two-observation case, it is clear from Eq. (3) that B has rank at most 2, and therefore

$$\det B = s_1 s_2 s_3 = 0 \tag{7}$$

Equations (6) and (7) show that in this case

$$s_3 = 0 \tag{8}$$

which leads to significant simplifications. The general n-observation case is treated in Markley [12]. Because $s_3 = 0$, we are free to choose the sign of the last column of U and of V so that both of these matrices have positive determinants. We shall assume that this is the case. Now,

$$\mathrm{trace}\,(AB^T) = \mathrm{trace}(AVSU^T) = \mathrm{trace}(WS) = s_1 W_{11} + s_2 W_{22} \tag{9}$$

where we have defined

$$W \equiv U^T AV \tag{10}$$

and have again used the invariance of the trace under cyclic permutations. W must be a proper orthogonal matrix, and so it is easy to see that the right side of Eq. (9) is maximized for $W_{11} = W_{22} = 1$, which means that W is the identity matrix. Thus, the optimal attitude is given by

$$A_{\mathrm{opt}} = UV^T \tag{11}$$

Equation (9) shows that this maximum is unique unless $s_2 = 0$. The vanishing of s_2 is the sign that the observations are not sufficient to determine the attitude. We shall see that this is related to the parallelism of the reference frame or body frame vectors.

Explicit Expression for the Optimal Attitude Matrix

The n-observation FOAM algorithm [15], specialized to the two-observation case, leads to an explicit expression for the optimal attitude matrix. We note that the classical adjoint, or adjugate (the transposed matrix of cofactors), of B^T is given in terms of the SVD by

$$\text{adj}\, B^T = U[\text{diag}(0, 0, s_1 s_2)]V^T \tag{12}$$

and that

$$BB^T B = U\Big[\text{diag}\Big(s_1^3, s_2^3, 0\Big)\Big]V^T \tag{13}$$

These allow us to write

$$\lambda s_1 s_2 A_{\text{opt}} = \lambda s_1 s_2 U V^T = (\lambda^2 - s_1 s_2)B - BB^T B + \lambda \text{adj} B^T \tag{14}$$

where

$$\lambda \equiv s_1 + s_2 = \text{trace}(A_{\text{opt}} B^T) \tag{15}$$

The parameter λ can be recognized as the maximum eigenvalue of Davenport's K matrix [2,3,9,15].

Direct computation from Eq. (3) gives

$$\text{adj}\, B^T = a_1 a_2 (\mathbf{b}_1 \times \mathbf{b}_2)(\mathbf{r}_1 \times \mathbf{r}_2)^T = a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2| \mathbf{b}_\times \mathbf{r}_\times^T \tag{16}$$

where $\mathbf{r}_\times$ and $\mathbf{b}_\times$ denote the normalized cross products

$$\mathbf{r}_\times \equiv (\mathbf{r}_1 \times \mathbf{r}_2)/|\mathbf{r}_1 \times \mathbf{r}_2| \tag{17a}$$

$$\mathbf{b}_\times \equiv (\mathbf{b}_1 \times \mathbf{b}_2)/|\mathbf{b}_1 \times \mathbf{b}_2| \tag{17b}$$

We note that $\mathbf{r}_\times$ or $\mathbf{b}_\times$ is undefined if the reference vectors or the observed vectors, respectively, are parallel or antiparallel. We see from Eqs. (3–5) that

$$s_1^2 + s_2^2 = \text{trace}(BB^T) = a_1^2 + a_2^2 + 2a_1 a_2 (\mathbf{b}_1 \cdot \mathbf{b}_2)(\mathbf{r}_1 \cdot \mathbf{r}_2) \tag{18}$$

and from Eqs. (12) and (16) that

$$s_1 s_2 = \{\text{trace}[(\text{adj} B^T)(\text{adj} B)]\}^{1/2} = a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2| \tag{19}$$

Equation (19) shows that $s_2 = 0$ if either of the cross products vanishes, which is the aforementioned unobservability condition. Now Eqs. (15), (18), and (19) give

$$\begin{aligned}\lambda^2 &= a_1^2 + a_2^2 + 2a_1 a_2[(\mathbf{b}_1 \cdot \mathbf{b}_2)(\mathbf{r}_1 \cdot \mathbf{r}_2) + |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|] \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos(\theta_b - \theta_r)\end{aligned} \tag{20}$$

where

$$\theta_b \equiv \cos^{-1}(\mathbf{b}_1 \cdot \mathbf{b}_2) \tag{21a}$$

$$\theta_r \equiv \cos^{-1}(\mathbf{r}_1 \cdot \mathbf{r}_2) \tag{21b}$$

In the two-observation case, λ is just the positive square root of Eq. (20), which was derived by a different method in Shuster and Oh [2]. If $\theta_b = \theta_r$, which would hold in the absence of measurement errors, this reduces to $\lambda = a_1 + a_2$. Finding λ in the case of more than two observations is much more difficult; it requires solving a quartic equation [2,9,15].

To complete the analytic derivation, we need to evaluate

$$\begin{aligned}BB^T B &= a_1^2[a_1 + a_2(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{b}_1 \cdot \mathbf{b}_2)]\mathbf{b}_1 \mathbf{r}_1^T \\ &\quad + a_2^2[a_1(\mathbf{r}_1 \cdot \mathbf{r}_2)(\mathbf{b}_1 \cdot \mathbf{b}_2) + a_2]\mathbf{b}_2 \mathbf{r}_2^T + a_1 a_2[a_1(\mathbf{b}_1 \cdot \mathbf{b}_2) \\ &\quad + a_2(\mathbf{r}_1 \cdot \mathbf{r}_2)]\mathbf{b}_1 \mathbf{r}_2^T + a_1 a_2[a_1(\mathbf{r}_1 \cdot \mathbf{r}_2) + a_2(\mathbf{b}_1 \cdot \mathbf{b}_2)]\mathbf{b}_2 \mathbf{r}_1^T\end{aligned} \tag{22}$$

and then, after some cancellations and rearrangement of terms,

$$\begin{aligned}(\lambda^2 - s_1 s_2)B - BB^T B &= a_1^2 a_2\{|\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|\mathbf{b}_1 \mathbf{r}_1^T \\ &\quad + [\mathbf{b}_2 - (\mathbf{b}_1 \cdot \mathbf{b}_2)\mathbf{b}_1][\mathbf{r}_2 - (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_1]^T\} \\ &\quad + a_1 a_2^2\{|\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|\mathbf{b}_2 \mathbf{r}_2^T \\ &\quad + [\mathbf{b}_1 - (\mathbf{b}_1 \cdot \mathbf{b}_2)\mathbf{b}_2][\mathbf{r}_1 - (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_2]^T\} \\ &= a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|\Big\{a_1\Big[\mathbf{b}_1 \mathbf{r}_1^T + (\mathbf{b}_1 \times \mathbf{b}_\times)(\mathbf{r}_1 \times \mathbf{r}_\times)^T\Big] \\ &\quad + a_2\Big[\mathbf{b}_2 \mathbf{r}_2^T + (\mathbf{b}_2 \times \mathbf{b}_\times)(\mathbf{r}_2 \times \mathbf{r}_\times)^T\Big]\Big\}\end{aligned} \tag{23}$$

Combining all of these intermediate results gives the final equation for the optimal attitude estimate, which first appeared in Markley [16],

$$\begin{aligned}A_{\text{opt}} &= (a_1/\lambda)\Big[\mathbf{b}_1 \mathbf{r}_1^T + (\mathbf{b}_1 \times \mathbf{b}_\times)(\mathbf{r}_1 \times \mathbf{r}_\times)^T\Big] \\ &\quad + (a_2/\lambda)\Big[\mathbf{b}_2 \mathbf{r}_2^T + (\mathbf{b}_2 \times \mathbf{b}_\times)(\mathbf{r}_2 \times \mathbf{r}_\times)^T\Big] + \mathbf{b}_\times \mathbf{r}_\times^T\end{aligned} \tag{24}$$

This solution has the advantage over other closed-form solutions of the two-observation Wahba problem [2,17,18] of explicitly exhibiting the dependence of the attitude estimate on the body frame and reference frame vectors. Thus, for example, it is obvious from Eq. (24) that the optimal attitude estimate maps $\mathbf{r}_1$ and $\mathbf{r}_2$ into the plane spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$. It is also easy to see that Eq. (24) has a unique limit as either a_1 or a_2 goes to zero, with λ equal to the nonzero weight in the limit. This is true even though Wahba's loss function of Eq. (1) does not have a unique minimum in either limit, because it effectively includes only one observation. In fact, as was first noted in Shuster and Oh [2], the solution of Wahba's problem reduces to a TRIAD estimate in either limit:

$$\begin{aligned}A_{\text{opt}} &= A_{\text{TRIAD-}I} \\ &\equiv [\,\mathbf{b}_1 \quad \mathbf{b}_\times \quad \mathbf{b}_1 \times \mathbf{b}_\times\,][\,\mathbf{r}_1 \quad \mathbf{r}_\times \quad \mathbf{r}_1 \times \mathbf{r}_\times\,]^T \quad \text{if } a_2 = 0\end{aligned} \tag{25a}$$

$$\begin{aligned}A_{\text{opt}} &= A_{\text{TRIAD-}II} \\ &\equiv [\,\mathbf{b}_2 \quad \mathbf{b}_\times \quad \mathbf{b}_2 \times \mathbf{b}_\times\,][\,\mathbf{r}_2 \quad \mathbf{r}_\times \quad \mathbf{r}_2 \times \mathbf{r}_\times\,]^T \quad \text{if } a_1 = 0\end{aligned} \tag{25b}$$

The general solution can be written as [19]

$$A_{\text{opt}} = \lambda^{-1}\Big[a_1 A_{\text{TRIAD-}I} + a_2 A_{\text{TRIAD-}II} + (\lambda - a_1 - a_2)\mathbf{b}_\times \mathbf{r}_\times^T\Big] \tag{26}$$

This form displays the relation between the optimal and TRIAD estimates, but is less efficient for computation than Eq. (24).

Covariance Analysis

Covariance estimates for the optimal and TRIAD attitude estimates have been known for a long time [2], but having an explicit form for the optimal attitude matrix permits a more complete covariance analysis for the two-observation case. We only consider errors in the measured body frame vectors, which are the most significant errors in practice. Including errors in the reference vectors would complicate the derivation without affecting the essential results [2,12]. We compute the covariance about the true values of the observed vectors, denoted by the subscript t, assuming that

$$\mathbf{b}_i = \mathbf{b}_{it} + \delta\mathbf{b}_i \quad \text{for } i = 1, 2 \tag{27}$$

Noting that $|\mathbf{b}_1 \times \mathbf{b}_2| = [1 - (\mathbf{b}_1 \cdot \mathbf{b}_2)^2]^{1/2}$, we find the first order variation of Eq. (20) to be

$$2\lambda\delta\lambda = 2a_1a_2(\delta\mathbf{b}_1 \cdot \mathbf{b}_{2t} + \mathbf{b}_{1t} \cdot \delta\mathbf{b}_2) \times [\mathbf{r}_1 \cdot \mathbf{r}_2 - (\mathbf{b}_{1t} \cdot \mathbf{b}_{2t})|\mathbf{b}_{1t} \times \mathbf{b}_{2t}|^{-1}|\mathbf{r}_1 \times \mathbf{r}_2|] = 0 \tag{28}$$

The final equality follows from $\mathbf{b}_{1t} \cdot \mathbf{b}_{2t} = \mathbf{r}_1 \cdot \mathbf{r}_2$ and $|\mathbf{b}_{1t} \times \mathbf{b}_{2t}| = |\mathbf{r}_1 \times \mathbf{r}_2|$ and shows that λ is unchanged to the first order from its value $\lambda = a_1 + a_2$ at the true values of the observed and measured vectors. Now from Eq. (24)

$$\delta A_{\text{opt}} = (a_1/\lambda)\Big[\delta\mathbf{b}_1\mathbf{r}_1^T + (\delta\mathbf{b}_1 \times \mathbf{b}_{\times t} + \mathbf{b}_{1t} \times \delta\mathbf{b}_\times)(\mathbf{r}_1 \times \mathbf{r}_\times)^T + \delta\mathbf{b}_\times\mathbf{r}_\times^T\Big] + (a_2/\lambda)\Big[\delta\mathbf{b}_2\mathbf{r}_2^T + (\delta\mathbf{b}_2 \times \mathbf{b}_{\times t} + \mathbf{b}_{2t} \times \delta\mathbf{b}_\times)(\mathbf{r}_2 \times \mathbf{r}_\times)^T + \delta\mathbf{b}_\times\mathbf{r}_\times^T\Big] \tag{29}$$

where $\delta\mathbf{b}_\times$ is implicitly defined by

$$\mathbf{b}_{\times t} + \delta\mathbf{b}_\times = [(\mathbf{b}_{1t} + \delta\mathbf{b}_1) \times (\mathbf{b}_{2t} + \delta\mathbf{b}_2)]/|(\mathbf{b}_{1t} + \delta\mathbf{b}_1) \times (\mathbf{b}_{2t} + \delta\mathbf{b}_2)| \tag{30}$$

The attitude estimation errors are most conveniently represented by a rotation angle vector in the body frame, given by

$$[\boldsymbol{\theta}\times] \equiv \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} = -(\delta A_{\text{opt}})A_t^T \tag{31}$$

The relations $A_t\mathbf{r}_i = \mathbf{b}_{it}$ then give

$$[\boldsymbol{\theta}\times] = -(a_1/\lambda)\Big[\delta\mathbf{b}_1\mathbf{b}_{1t}^T + (\delta\mathbf{b}_1 \times \mathbf{b}_{\times t} + \mathbf{b}_{1t} \times \delta\mathbf{b}_\times)(\mathbf{b}_{1t} \times \mathbf{b}_{\times t})^T + \delta\mathbf{b}_\times\mathbf{b}_{\times t}^T\Big] - (a_2/\lambda)\Big[\delta\mathbf{b}_2\mathbf{b}_{2t}^T + (\delta\mathbf{b}_2 \times \mathbf{b}_{\times t} + \mathbf{b}_{2t} \times \delta\mathbf{b}_\times)(\mathbf{b}_{2t} \times \mathbf{b}_{\times t})^T + \delta\mathbf{b}_\times\mathbf{b}_{\times t}^T\Big] \tag{32}$$

The error $\delta\mathbf{b}_i$ is perpendicular to $\mathbf{b}_{it}$ to the first order in the errors, and so we have to the same order

$$\delta\mathbf{b}_1 = (\mathbf{b}_{\times t} \cdot \delta\mathbf{b}_1)\mathbf{b}_{\times t} - |\mathbf{b}_{1t} \times \mathbf{b}_{2t}|^{-1}(\mathbf{b}_{2t} \cdot \delta\mathbf{b}_1)(\mathbf{b}_{1t} \times \mathbf{b}_{\times t}) \tag{33a}$$

$$\delta\mathbf{b}_2 = (\mathbf{b}_{\times t} \cdot \delta\mathbf{b}_2)\mathbf{b}_{\times t} + |\mathbf{b}_{1t} \times \mathbf{b}_{2t}|^{-1}(\mathbf{b}_{1t} \cdot \delta\mathbf{b}_2)(\mathbf{b}_{2t} \times \mathbf{b}_{\times t}) \tag{33b}$$

$$\delta\mathbf{b}_\times = |\mathbf{b}_{1t} \times \mathbf{b}_{2t}|^{-1}[(\mathbf{b}_{\times t} \cdot \delta\mathbf{b}_2)(\mathbf{b}_{1t} \times \mathbf{b}_{\times t}) - (\mathbf{b}_{\times t} \cdot \delta\mathbf{b}_1)(\mathbf{b}_{2t} \times \mathbf{b}_{\times t})] \tag{33c}$$

Inserting these into Eq. (32) gives, after many vector and matrix manipulations including applications of the identity $[(\mathbf{u} \times \mathbf{v})\times] = \mathbf{v}\mathbf{u}^T - \mathbf{u}\mathbf{v}^T$,

$$\boldsymbol{\theta} = |\mathbf{b}_1 \times \mathbf{b}_2|^{-1}\Big\{\Big[\mathbf{b}_2\mathbf{b}_\times^T - (a_1/\lambda)\mathbf{b}_\times\mathbf{b}_2^T\Big]\delta\mathbf{b}_1 - \Big[\mathbf{b}_1\mathbf{b}_\times^T - (a_2/\lambda)\mathbf{b}_\times\mathbf{b}_1^T\Big]\delta\mathbf{b}_2\Big\} \tag{34}$$

We have omitted the subscript t denoting true values in this equation and in the remainder of this section to simplify the notation. In practice, the true values are not known, and the covariance is generally computed using the measured vectors. Because $\delta\mathbf{b}_i$ is assumed to be zero mean, ignoring higher-order corrections required by the unit-vector norm constraint, we see that the error angle vector has zero mean, that is, that the estimate is unbiased.

We define the measurement covariances by

$$R_i \equiv E[\delta\mathbf{b}_i\delta\mathbf{b}_i^T] \quad \text{for } i = 1, 2 \tag{35}$$

and assume that the measurement errors are independent, $E[\delta\mathbf{b}_1\delta\mathbf{b}_2^T] = 0$. Then the covariance matrix of the attitude error, defined as the expectation $P \equiv E[\boldsymbol{\theta}\boldsymbol{\theta}^T]$, is

$$P = |\mathbf{b}_1 \times \mathbf{b}_2|^{-2}\Big\{\Big[\mathbf{b}_2\mathbf{b}_\times^T - (a_1/\lambda)\mathbf{b}_\times\mathbf{b}_2^T\Big]R_1\Big[\mathbf{b}_2\mathbf{b}_\times^T - (a_1/\lambda)\mathbf{b}_\times\mathbf{b}_2^T\Big]^T + \Big[\mathbf{b}_1\mathbf{b}_\times^T - (a_2/\lambda)\mathbf{b}_\times\mathbf{b}_1^T\Big]R_2\Big[\mathbf{b}_1\mathbf{b}_\times^T - (a_2/\lambda)\mathbf{b}_\times\mathbf{b}_1^T\Big]^T\Big\} \tag{36}$$

Consider the case in which the measurement errors are isotropically distributed around the measured vector (the QUEST measurement model [2]),

$$R_i = \sigma_i^2\Big(I_{3\times3} - \mathbf{b}_i\mathbf{b}_i^T\Big) \quad \text{for } i = 1, 2 \tag{37}$$

In this case, the covariance simplifies to

$$P = [(a_1\sigma_1/\lambda)^2 + (a_2\sigma_2/\lambda)^2]\mathbf{b}_\times\mathbf{b}_\times^T + |\mathbf{b}_1 \times \mathbf{b}_2|^{-2}\Big(\sigma_1^2\mathbf{b}_2\mathbf{b}_2^T + \sigma_2^2\mathbf{b}_1\mathbf{b}_1^T\Big) \tag{38}$$

We can choose the weights a_1 and a_2 to minimize this covariance. Setting $\partial P/\partial a_i = 0$ for $i = 1$ or 2 gives the condition

$$\sigma_1^2 a_1 = \sigma_2^2 a_2 \tag{39}$$

which is the well-known condition that the weights be proportional to the inverse measurement variances [2]. With these weights, the covariance takes the well-known form [15]

$$P = \sigma_1^2\sigma_2^2\Big(\sigma_1^2 + \sigma_2^2\Big)^{-1}\mathbf{b}_\times\mathbf{b}_\times^T + |\mathbf{b}_1 \times \mathbf{b}_2|^{-2}\Big(\sigma_1^2\mathbf{b}_2\mathbf{b}_2^T + \sigma_2^2\mathbf{b}_1\mathbf{b}_1^T\Big) \tag{40}$$

It is interesting to find the weights that minimize the trace of the attitude covariance matrix for general measurement covariances, which can provide a better measurement model for sensors with large fields of view [20]. Taking the trace of Eq. (36) and using the cyclic permutation property of the trace and some elementary matrix/vector algebra gives

$$\text{trace}P = |\mathbf{b}_1 \times \mathbf{b}_2|^{-2}\Big[\mathbf{b}_\times^T(R_1 + R_2)\mathbf{b}_\times + (a_1/\lambda)^2\mathbf{b}_2^TR_1\mathbf{b}_2 + (a_2/\lambda)^2\mathbf{b}_1^TR_2\mathbf{b}_1\Big] \tag{41}$$

Setting $\partial(\text{trace}P)/\partial a_i = 0$ for $i = 1$ or 2 gives a condition analogous to Eq. (39):

$$\Big(\mathbf{b}_2^TR_1\mathbf{b}_2\Big)a_1 = \Big(\mathbf{b}_1^TR_2\mathbf{b}_1\Big)a_2 \tag{42}$$

We see that in this case the optimal weights depend on the measurement geometry. It should be pointed out that Wahba's optimality condition does not have as secure a basis in this case as it does for the QUEST measurement model [21].

Conclusions

This paper has presented a new analysis of an old problem. The solution for the optimal attitude matrix is equivalent to previously published results, but the final form of Eq. (24) is simpler than any other in the literature. The covariance matrix for general measurement errors given by Eq. (36) is believed to be new, and the specialization to the case of isotropic measurement errors, although not new, is also simpler than earlier expressions.

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